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On minimum maximal independent sets of a graph

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Abstract

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For a simple graph G, the *independent domination number* $i(G)$ is defined to be the minimum cardinality among all maximal independent sets of vertices of G. If G has order n and minimum degree $\delta \leqslant n/2$, we give upper bounds for $i(G)$ as functions of n and δ, and over part of the range achieve best possible results. In particular, we extend work of Favaron (1988).

1. Introduction

Let $G=(V, E)$ be a simple graph of order $|V|=n$ and minimum degree δ. An *independent set* is a set of pairwise non-adjacent vertices of G. The *independent domination number* $i(G)$ is defined to be the minimum cardinality among all maximal independent sets of G. Equivalently, it is the minimum cardinality among all cliques in the complement $\bar{G}$. This parameter was introduced in [3]; note that $i(G)$ gives a lower bound on the number of vertices receiving the first colour via the greedy colouring algorithm.

Several previous papers on the subject [2, 4, 7] have been concerned with bounding the sum of $i(G)$ and other graph parameters such as $\beta(G)$, the *independence number* (the maximum cardinality taken over all independent sets of G). For example, in [2] it was shown that

$$i(G)+\beta(G) \leqslant 2n+2\delta-2\sqrt{2n\delta}.$$

Up to this point, little work had been done purely on $i(G)$ itself. However, in [5] Favaron was able to make improvements upon this last inequality, from which she then recovered the upper bounds for $i(G)$ in terms of n and δ stated in Proposition 1 below.

In what follows, we abbreviate $i(G)$ to i where it is unambiguous. Then *open neighbourhood* of a vertex v will be denoted by $\Gamma(v) = \{u \in V: uv \in E\}$, and that of a set of vertices X by $\Gamma(X) = \bigcup_{x \in X} \Gamma(x) \cap (V - X)$.

Proposition 1 (Favaron [5]). *In any graph of order n and minimum degree δ,*

$$i \leq \min\{n - \delta, n + 3\delta - 2\sqrt{\delta(n + 2\delta)}\}.$$

The first bound applies for $n/2 \leq \delta \leq n$, and it is easily checked (by Proposition 2) that this is attained only by complete multipartite graphs with vertex classes all of the same order. In contrast, when $\delta \leq n/2$ there is much scope for tightening the inequality. (The curve $i = n + 3\delta - 2\sqrt{\delta(n + 2\delta)}$ is shown in the figure below, as are other curves relating to this paper). This fact was noted in [5], where the next conjecture was stated.

Conjecture (Favaron [5]). In any graph of order n and minimum degree δ,

$$i \leq n + 2\delta - 2\sqrt{n\delta}.$$

Favaron observed that by a theorem of Bollobás and Cockayne in [1], the conjecture was correct for the case $\delta = 1$. She also cited this next class of graphs which would be extremal if the conjecture were to be true in general. For δ and l positive integers, let $F(\delta, l)$ be the family of graphs such that $V = \bigcup_{i=1}^{l} (S_i \cup F_i)$, where $|F_i| = \delta$, $|S_i| = \delta(l - 1)$ and $xy \in E$ if and only if $x \in S_i, y \in F_i$ or $x \in F_i, y \in F_j, i \neq j$. Then the graphs of $F(\delta, l)$ satisfy $i = n + 2\delta - 2\sqrt{n\delta}$.

In this paper, we show how the bounds of both Proposition 1 and the conjecture can be substantially sharpened when $n/4 < \delta < n/2$, and over part of the range achieve best possible results. Although we have not managed to establish the conjecture for $\delta \leq n/4$, we do obtain Theorem 7, which is noticeably stronger than Proposition 1. Moreover a simplified version of the proof of Theorem 7 yields a straightforward proof of Proposition 1 (details are given in [6]); Favaron's proof was somewhat indirect.

2. Results

We start with an elementary proposition employed regularly in later proofs.

Proposition 2. *In any graph of order n and maximum degree Δ, $i \leq n - \Delta$.*

Proof. Take a vertex of degree Δ and extend to a maximal independent set. □

Henceforth we denote by Δ the maximum degree of a graph under discussion. Our first improvement upon previously known results is given by the following lemma.

Lemma 3. *Let G be a graph of order n and minimum degree δ, and let I be a minimum maximal independent set of G. If no vertex of $V - I$ is joined to all of I, then $i \leqslant n - \sqrt{n\delta}$.*

Proof. Choose $x \in V - I$ such that $k = |\Gamma(x) \cap I|$ is maximal, and let $K = \Gamma(x) \cap I$. Form the set $X = \{v \in V - I : \Gamma(v) \cap I \subseteq K\}$, and let R be a maximal independent set of $G[X]$ containing x. Then $R \cup (I - K)$ is maximal independent for G, so $|R| + (i - k) \geqslant i$, implying $|X| \geqslant |R| \geqslant k$. The average degree of a vertex of $V - I$ in I is at least $i\delta/(n - i)$, giving

$$|X| \geqslant |R| \geqslant k \geqslant i\delta/(n - i). \tag{1}$$

Now $I - K \neq \emptyset$ by the conditions of the lemma. Also $\Gamma(I - K) \subseteq V - I - X$, with each vertex of $I - K$ having at least δ neighbours in $V - I - X$, so

$$|V - I - X| \geqslant \delta. \tag{2}$$

Combining (1) and (2) we have

$$n - i = |X| + |V - I - X| \geqslant i\delta/(n - i) + \delta;$$

solving the resultant quadratic expression gives $i \leqslant n - \sqrt{n\delta}$. □

Corollary 4. *In any graph of order n and minimum degree $\delta \geqslant n/4$, $i \leqslant n/2$.*

Proof. We may assume $n/4 \leqslant \delta < n/2$, since for $\delta \geqslant n/2$ Proposition 1 implies $i \leqslant n - \delta \leqslant n/2$. Suppose $i > n/2$ for some graph G, and let I be a minimum maximal independent set of G. By Proposition 2, $\Delta \leqslant n - i < i$, so no vertex of $V - I$ can be joined to all of I. Therefore Lemma 3 applies, and $i \leqslant n - \sqrt{n\delta} \leqslant n/2$, a contradiction. □

In addition to the corollary, Lemma 3 yields a better upper bound for i than Proposition 1 for $3(7 - 4\sqrt{3})n = 0.215 \cdots n < \delta \leqslant n/4$, although later we obtain even stronger inequalities. For the moment, we turn our attention to improving upon Corollary 4 when $n/4 < \delta < n/2$.

Theorem 5. *In any graph of order n and minimum degree δ, if $n/4 \leqslant \delta \leqslant 2n/5$ then $i \leqslant 2(n - \delta)/3$, and if $2n/5 \leqslant \delta \leqslant n/2$ then $i \leqslant \delta$.*

Proof. Suppose some graph G with $n/4 \leqslant \delta \leqslant n/2$ has a minimum maximal independent set I of order i. If $i \leqslant \max\{\delta, n - \sqrt{n\delta}\}$, then the conclusion of the theorem holds, so we may assume $i > \max\{\delta, n - \sqrt{n\delta}\}$. Let $S = \{v \in V - I : \Gamma(v) \supseteq I\}$. As $i > n - \sqrt{n\delta}$, by Lemma 3 we have $S \neq \emptyset$. Further, if $S = V - I$ then this would imply that all vertices of G had degree greater than δ; we conclude that $V - I - S \neq \emptyset$.

Let $T = \{v \in V - I - S : \Gamma(v) \supseteq S\}$. We know $T \neq V - I - S$, else we can find a maximal independent set of $V - I$ from S which is also maximal independent for G. The minimality of I would imply $|S| \geq |I| > \delta$; but this means that all vertices of G have degree greater than δ, an obvious contradiction. Hence $V - I - S - T \neq \emptyset$.

Suppose $|\Gamma(v) \cap (V - I)| > n - 2i$ for some $v \in V - I - S - T$. Then we could find a maximal independent set I^* of $V - I$, including v and one of its nonneighbours in S, which was also maximal independent for G. Since $\Gamma(v) \cap I^* = \emptyset$, we have $|I^*| < (n - i) - (n - 2i) = i$, contradicting the minimality of I. Thus for all $v \in V - I - S - T$ we know

$$|\Gamma(v) \cap (V - I)| \leq n - 2i. \tag{1}$$

For any $x \in V - I - S - T$, let $K = \Gamma(x) \cap I$ and $|K| = k$. As $|\Gamma(x)| \geq \delta$, by (1) we have $k \geq \delta - n + 2i$. Form the set $X = \{v \in V - I : \Gamma(v) \cap I \subseteq K\}$ and let R be a maximal independent set of $G[X]$ containing x. Then $R \cup (I - K)$ is maximal independent for G, so $|R| + (i - k) \geq i$, implying

$$|X| \geq |R| \geq k \geq \delta - n + 2i. \tag{2}$$

Since $x \notin S$ we know $I - K \neq \emptyset$. Also $\Gamma(I - K) \subseteq V - I - X$, with each vertex of $I - K$ having at least δ neighbours in $V - I - X$, so

$$|V - I - X| \geq \delta. \tag{3}$$

Combining (2) and (3) gives

$$n - i = |X| + |V - I - X| \geq (\delta - n + 2i) + \delta,$$

and rearranging we have $i \leq 2(n - \delta)/3$. The proof is completed by observing that both $2(n - \delta)/3 \leq \delta$ and $n - \sqrt{n\delta} < \delta$ hold for $\delta \geq 2n/5$. □

Note that Theorem 5 is best possible for $2n/5 \leq \delta \leq n/2$, as the complete bipartite graphs $K_{\delta, n-\delta}$ satisfy $i = \delta$. The same is not true for $n/4 < \delta < 2n/5$; in [6] we demonstrate, by involved arguments, that $i < 2(n - \delta)/3$ strictly for δ in this range. No doubt by putting in more work we could strengthen this bound further, but probably a fresh idea is required in order to get a best possible result. We do however give the following lower bound.

Theorem 6. *For each rational number $p/q \in [\frac{1}{4}, \frac{3}{8}]$ there is a graph G of order n with minimum degree $\delta = pn/q$ and $i = 3n/4 - \delta$.*

Proof. For a, b positive integers, let $H(a, b)$ be the family of graphs such that $V = \bigcup_{i=0}^{3} (A_i \cup B_i)$, where $|A_i| = a$, $|B_i| = b$ and $xy \in E$ if and only if $x \in A_i$, $y \in B_i$ or $x \in A_{i+1 \pmod 4}$, $y \in B_i$ or $x \in B_i$, $y \in B_{i+2 \pmod 4}$ or $x \in A_i$, $y \in A_j$, $i \neq j$. Then the graphs $H(a, b)$ have order $n = 4(a + b)$, minimum degree $\delta = 2a + b$, and satisfy $i = 3n/4 - \delta$. Take $a = 4p - q$, $b = 2q - 4p$. □

We suspect that the graphs of Theorem 6 are best possible for $n/4 \leq \delta \leq 3n/8$, and also that $i \leq \delta$ for $3n/8 \leq \delta \leq 2n/5$, although both of these are still open problems.

In our final theorem, we show how to obtain sharper upper bounds for i than those of Proposition 1 and Lemma 3 when $\delta \leq n/4$.

Theorem 7. *In any graph of order n and minimum degree δ, if $0 \leq \delta \leq (n-2)/7$ then*

$$i \leq n + 3\delta - \min\{1 + 2\sqrt{\delta(n + 2\delta - 2)}, 2\sqrt{\delta(n + 9\delta/4)}\},$$

and if $(n-2)/7 \leq \delta \leq n/4$ then $i \leq 2(n-\delta)/3$.

Remark. Note that $2\sqrt{\delta(n+9\delta)/4)} < 1 + 2\sqrt{\delta(n + 2\delta - 2)}$ only if $\delta = O(n^{\frac{1}{3}})$.

Proof. Suppose the theorem is false for some graph G; that is

either $0 \leq \delta \leq (n-2)/7$ and $i > \max\{A, B\}$,

or $(n-2)/7 \leq \delta \leq n/4$ and $i > C$,

where $A = n + 3\delta - 1 - 2\sqrt{\delta(n + 2\delta - 2)}$, $B = n + 3\delta - \sqrt{\delta(4n + 9\delta)}$ and $C = 2(n-\delta)/3$.

Let $k_0 = 1 + \sqrt{(i-1)\delta}$ and $k_1 = n - i - \delta$. Prior to giving the main argument of the proof, we state the following facts, all of which are easily verified.

(F_1) The hypothesis implies $i > n/2$ and $i > n + 2\delta - 2\sqrt{\delta(n+\delta)}$.

(F_2) We have $C \geq B$ if and only if $n/10 \leq \delta \leq n/4$.

(F_3) If $i > C$ then $k_0 > 1 + \sqrt{(2n - 2\delta - 3)\delta/3}$ and $k_1 < (n-\delta)/3$; so if in addition $(n-2)/7 \leq \delta \leq n/4$, observe that $k_1 < k_0$.

Continuing with the proof, we choose a minimum maximal independent set I of G and $x \in V - I$ such that $k = |\Gamma(x) \cap I|$ is maximal. From Proposition 2 and (F_1), we have $k \leq \Delta \leq n - i < i$. Let $K = \Gamma(x) \cap I$. Then constructing the sets X and R as in the proof of Lemma 3, we deduce that $R \cup (I - K)$ is maximal independent for G and $|X| \geq |R| \geq k$. Therefore

$$n - i = |X| + |V - I - X| \geq k + |V - I - X|. \tag{1}$$

Firstly, consider the case $k \leq n - i - k$. Now $\Gamma(I - K) \subseteq V - I - X$, with each member of $I - K$ having degree at least δ. Moreover, each vertex of $V - I - X$ has degree at most k in $I - K$, so substituting in (1) we get

$$n - i \geq k + (i-k)\delta/k. \tag{2}$$

As a function of k, the right-hand side of (2) minimises at $k = \sqrt{i\delta}$; but $\sqrt{i\delta} \geq (n-i)/2$ for $i \geq n + 2\delta - 2\sqrt{\delta(n+\delta)}$, which we noted to hold in (F_1). Hence in fact the right-hand side of (2) minimises at $k = (n-i)/2$, giving $i \leq B$.

If not, we must have $k > n - i - k$. If all vertices of $V - I - X$ have degree at most $n - i - k$ in $I - K$, then since each member of $I - K$ has at least δ

neighbours in $V-I-X$, substituting in (1) we obtain

$$n-i \geqslant k+(i-k)\delta/(n-i-k).$$

The right-hand side of this expression is an increasing function of k, and so minimises at $k=(n-i)/2$, again yielding $i \leqslant B$. Thus, in these first two cases, the hypothesis implies $(n-2)/7 \leqslant \delta \leqslant n/4$ and $i > C$; but this is in contradiction with (F_2).

Otherwise, some $x' \in V-I-X$ must have degree $k' > n-i-k$ in $I-K$. We form the sets K', X' and R' for x' in an analogous way to those formed for x, and deduce $|X'| \geqslant |R'| \geqslant k' > n-i-k$. Therefore $|R|+|R'| > n-i$, so $R \cap R' \neq \emptyset$. Now $\Gamma(R \cap R') \cap I \neq \emptyset$ by the maximality of I, and $\Gamma(R \cap R') \cap I \subseteq (K \cap K')$. Hence x' must have at least $|\Gamma(R \cap R')|$ neighbours in K and so at most $k-|\Gamma(R \cap R')|$ in $I-K$. In addition, each member of $I-K$ has at least δ neighbours in $V-I-X$, so substituting in (1) we have

$$n-i \geqslant k+(i-k)\delta/(k-|\Gamma(R \cap R')|) \geqslant k+(i-k)\delta/(k-1). \tag{3}$$

As a function of k, the right-hand side of (3) minimises at $k=k_0$, implying $i \leqslant A$; thus by the hypothesis we must have $(n-2)/7 \leqslant \delta \leqslant n/4$ and $i > C$. However, $|V-I-X| = |\Gamma(I-K)| \geqslant \delta$, which substituted in (1) gives $k \leqslant k_1$, and (F_3) shows further that $k_1 < k_0$. We conclude that in this final case the right-hand

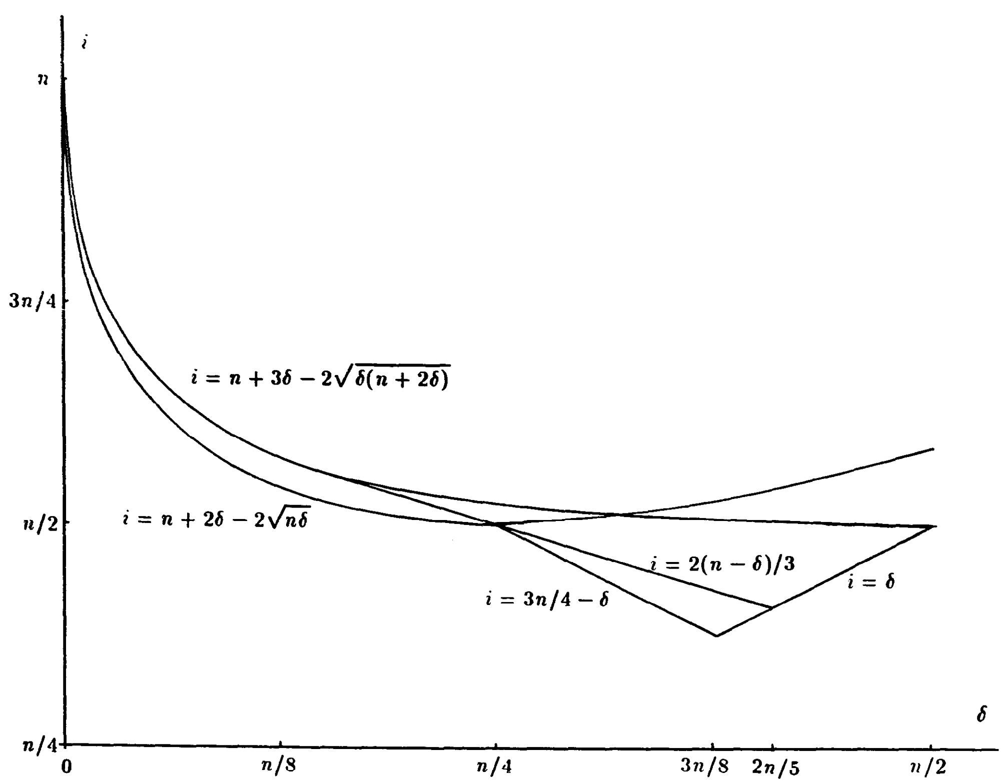

Fig. 1. The bounds on i as functions of n and δ.

side of (3) minimises at $k = k_1$, whence $i \leq C$. This contradiction completes the proof. □

Lastly, we note that the conjecture of Favaron may not be best possible in the range $0 < \delta < n/4$. In [6] we apply the argument of Theorem 6 to the graphs $F(\delta, l)$ and show that the tangents on the left at the points $\delta = n/l^2$ are lower bounds. We have been unable to find any graphs lying above these lines.

Fig. 1 shows the relationship between the various bounds on i featured in our results.

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